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ON THE PROPAGATION OF ELASTIC VIBRATIONS FROM A POINT SOURCE WITHIN AN ANISOTROPIC HALF-SPACE*

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There is investigated the solution, obtained in /1/, for the problem of elastic vibrations propation in an anisotropic half-space with four elastic constants, from a point source of an instantaneous pulse type. By using the results from /2--4/, the features of wave propagation as a function of the elastic constants relationships are studied. The case is examined, when the elastic constants satisfy conditions for which the wave fronts have no acute-angled edges. A Riemann surface is constructed for a single-valued determination of the solution, and the correspondence between points of this surface and the wave fields is studied. To determine the incident and reflected waves on one two-sheeted surface, there is introduced the concept of a two-layer domain of definition of the reflected waves. Expressions are obtained for the primary, propagated from the source, and secondary perturbation fronts reflected from the half-space boundary.

Wave process in anisotropic media are subject to more complex regularities than in isotropic media since the equations of motion of a medium do not reduce to wave equations. In contrast to the isotropic media with two types of waves (purely longitudinal and purely transverse) three kinds of waves (one quasi-longitudinal and two quasi-transverse) propagated at different velocities, can exist in anisotropic media. The displacement vectors in such waves have normal and tangential components to the front, and the propagation velocities depend on the direction. The wave propagation depends qualitatively on the relationship between the elastic constants. The examination of these questions is of definite difficulty, hence, the solutions obtained in /1/ have not yet been investigated completely.

Upon going from an anisotropic to an isotropic medium in the plane problem, the quasi-longitudinal wave goes over into a longitudinal, one of the two quasi-transverse waves goes over into a transverse SV wave with displacements in the plane of wave propagation, and the other goes over into a transverse SH wave with displacements perpendicular to the same plane. In contrast to the quasi-transverse waves, these transverse waves are propagated at an identical velocity comprising a single transverse wave. By analogy with the isotropic medium, we call the corresponding quasi-transverse waves SV and SH waves.

As in /1/, we limit ourselves to a study of wave propagation of the first type (quasi-longitudinal), and of the second type (quasi-transverse SV). These waves cannot exist separately. Study of the waves of the third kind (quasi-transverse of SH type) is trivial.

1. Let us examine the elastic vibrations in an anisotropic half-space $y \geqslant 0$ with four elastic constants, caused by a point source of the instantaneous pulse type that is at the point $x = 0, y = y_0$ at the time $t = 0$. We have analogous motion even in a medium with nine elastic constants if the coordinate planes coincide with the planes of elastic symmetry, and the vibrations are independent of the coordinate z.

The displacement vector components of the quasi-longitudinal and quasi-transverse SV type waves are determined by the expressions

$$u = \operatorname{Re} \sum_{k=1}^{2} U(\theta_k, \lambda_k, w_k(\zeta)), \quad v = \operatorname{Re} \sum_{k=1}^{2} V(\theta_k, \lambda_k, w_k(\zeta)) \tag{1.1}$$

$$U(\theta, \lambda, w(\zeta)) = c \int^{\theta} \zeta \lambda w(\zeta)\, d\zeta, \quad V(\theta, \lambda, w(\zeta)) = \int^{\theta} (a\zeta^2 + d\lambda^2 - 1)\, w(\zeta)\, d\zeta$$

The lower limits of the integrals are arbitrary; in particular, points at which the integrands are fixed /1/ can be taken as these limits.

The complex variables θ_k and the quantities λ_k are defined by the relationships

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$$1-\theta_k\xi+\lambda_k(\eta-\eta_0)=0 \quad (\xi=x/t;\ \eta=y/t;\ \eta_0=y_0/t) \tag{1.2}$$
$$\lambda_k=\left(\frac{[(a+b)-L\theta_k^2]+(-1)^k\sqrt{Q(\theta_k)}}{2bd}\right)^{1/2} \quad (k=1;2)$$
$$Q(\theta_k)=[(b+d)-L\theta_k^2]^2-4bd(1-a\theta_k^2)(1-d\theta_k^2),\quad L=ab+d^2-c^2$$

The functions λ_k are branches of the algebraic function λ which is unique on a Riemann surface whose form depends on the relationship between the elastic constants and is studied in /3/. The functions w_k are branches of an arbitrary analytic function w that is unique on a two-sheeted Riemann surface, which are selected so that /2,3/ their real parts would vanish on the edges of slits along the real axes of planes of the Riemann surface where the λ_k take on real values.

For real media of the anisotropy class under consideration, the ratios between the elastic constants and the density satisfy the conditions

$$a>d,\quad b>d,\quad d>0,\quad K_1=ab-(c-d)^2>0 \tag{1.3}$$
$$a=C_{11}/\rho,\quad b=C_{22}/\rho,\quad d=C_{66}/\rho,\quad c=(C_{66}+C_{12})/\rho$$

When these quantities satisfy the additional condition

$$K_2=ab-(c+d)^2<0 \tag{1.4}$$

two of the four branch points of the inner radical of λ_k

$$\theta_i^\circ=\left(\frac{M\pm\sqrt{-4bdc^2N_1}}{K_1K_2}\right)^{1/2} \tag{1.5}$$
$$M=(b+d)N_1-(b-d)(a-b)d,\quad N_1=(a-d)(b-d)-c^2$$

are real and two are imaginary.

Elastic wave propagation in media satisfying the condition (1.4) depends on the signs of the quantities

$$N_2=(a-d)b-c^2,\quad N_3=(b-d)a-c^2 \tag{1.6}$$

2. Let us consider the case when $N_2>0$ and $N_3>0$. According to /4/, the Riemann surface has the form displayed in Figs.1 and 2 (the shape of the omitted left sides of the surfaces is symmetric to that presented). The slit edges (θ_i°, ∞) of the planes θ_1 and θ_2 are glued crosswise. The functions λ_k are fixed on the planes θ_k so that they would be positive for $\theta_k=i\beta$, where β is a sufficiently small positive quantity. The functions λ_1 and λ_2 take on real values on the edges $(-1/\sqrt{a}, +1/\sqrt{a})$ of the θ_1 plane, and $(-1/\sqrt{d}, +1/\sqrt{d})$ of the θ_2 plane. On the edges of these slits, the real parts of the functions w_1 and w_2 vanish correspondingly.

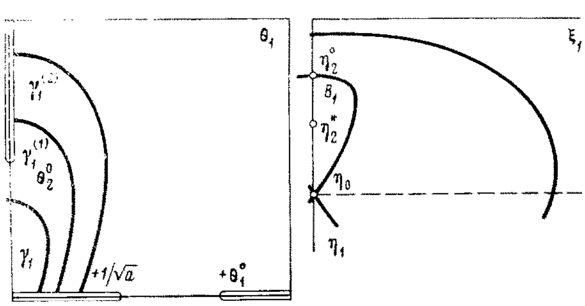

Fig.1

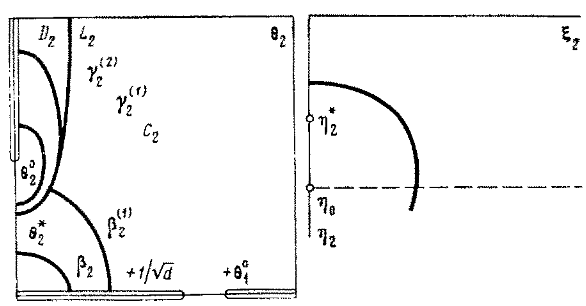

Fig.2

The correspondence between points of the planes θ_k and points of the plane $\xi\eta$ is expressed by (1.2), and can be established on the basis of the results in /4/. We shall consider the quasi-longitudinal wave to be propagated in the $\xi_1\eta_1$ plane (Fig.1), and the quasi-transverse wave in the plane $\xi_2\eta_2$ (Fig.2). The subscripts of the coordinates of points displaying affiliation with the planes $\xi_k\eta_k$ are not indicated in (1.2).

The front of quasi-lonitudinal and quasi-transverse waves with their semi-tangents correspond to the edges of the slits $(-1/\sqrt{a}, +1/\sqrt{a})$ of the θ_1 plane and $(-1/\sqrt{d}, +1/\sqrt{d})$

of the θ_2 plane. Terms of the solution (1.1) vanish at points of the front of the quasi-longitudinal wave and its external domain as $k = 1$; and at points of the quasi-transverse wave front and its external domain for $k = 2$. There are no vibrations in domains of the planes $\xi_k\eta_k$ external to the wave fronts.

Although the vibrations (1.1) do not reach the half-space boundary $\eta_k = 0$, the quasi-longitudinal and quasi-transverse wave fronts are convex closed curves /2/ and can be expressed as envelopes of the lines (1.2) for real values of θ_k and λ_k in the form

$$\xi_k = -\lambda_k' / (\lambda_k - \theta_k\lambda_k'), \quad \eta_k = -1 / (\lambda_k - \theta_k\lambda_k') + \eta_0 \tag{2.1}$$

The upper edges of the slits correspond the upper halves of the wave fronts $(\eta_k \leqslant \eta_0)$, and the lower to the lower half-planes $(\eta_k \geqslant \eta_0)$.

Analogously, the upper halves of the wave fields $(\eta_k < \eta_0)$ correspond to the upper half-planes θ_k, and the lower $(\eta_k > \eta_0)$ to the lower. The points $\xi_k = 0, \eta_k = \eta_0$ on the wave fields correspond to the infinitely remote points of the planes θ_k. The segments $(-\sqrt{a}, +\sqrt{a})$ and $(-\sqrt{d}, +\sqrt{d})$ cut off by the wave fronts on the horizontal axes of symmetry by the wave fields $\eta_1 = \eta_0$ and $\eta_2 = \eta_0$, correspond to the segments $(\pm 1 / \sqrt{a}, \pm\infty)$ and $(\pm 1 / \sqrt{d}, \pm\infty)$ of the real axes of the planes θ_1 and θ_2.

The functions λ_k take on complex values on the edges of the slits $(\theta_2^\circ, i\infty)$ of the planes θ_k. A line in the left (right) upper quarter of the quasi-longitudinal wave field corresponds to glueing the left (right) edge of the slit of the θ_1 plane and the right (left) edge of the slit of the θ_2 plane. The ends of these lines coincide at the points $\eta_1 = \eta_2^\circ = -1 / \lambda_k(\theta_2^\circ) + \eta_0$ and $\eta_1 = \eta_0$ of the coordinate axes forming a closed contour bounding the domain B_1 in the upper half of the quasi-longitudinal wave field that is symmetric relative to this axis. The rest of the upper half of the quasi-longitudinal wave field is denoted by A_1.

The functions λ_k have real values, where $\lambda_1(\theta_2^\circ) = \lambda_2(\theta_2^\circ)$, on the sections $(0, \theta_2^\circ)$ of the positive imaginary half-axes of the θ_1 and θ_2 planes. The sections $(\eta_0 - \sqrt{b}) \leqslant \eta_1 \leqslant \eta_2^\circ$ and $(\eta_0 - \sqrt{d}) \leqslant \eta_2 \leqslant \eta_2^*$, of the ordinate axes η_1 and η_2, set in correspondence to the expressions $\eta_k = \eta_0 - 1 / \lambda_k$, will correspond to the sections $0 \leqslant \theta_1 \leqslant \theta_2^\circ$ and $0 \leqslant \theta_2 \leqslant \theta_2^*$, where /4/

$$\theta_2^* = i\,[-(\sqrt{ad}M + L\sqrt{c^2[c^2 - (a-d)(b-d)]}) / (\sqrt{ad}K_1K_2)]^{1/2} \tag{2.2}$$

The section $\eta_2^\circ \leqslant \eta_1 \leqslant \eta_2^*$ of the η_1 ordinate axis, set in correspondence to the expression $\eta_1 = \eta_0 - 1/\lambda_2$ and belonging to the domain B_1, corresponds to the section $\theta_2^\circ \geqslant \theta_2 \geqslant \theta_2^*$ of the positive imaginary half-axis of the θ_2 plane.

The lines L_2 in the first and second quadrants of the θ_2 plane, going from the point $\theta_2^* = i\varepsilon_2^*$ to the infinitely remote point and expressed by the functions /4/

$$\begin{aligned} \theta_2 &= \pm\,[(A + \sqrt{A^2 - B}) / (\sqrt{ad}K_1K_2)]^{1/2} + i\varepsilon_2 \\ A &= \sqrt{ad}\,[M - (L^2 + 4abd^2)\varepsilon_2^2] \\ B &= K_1K_2\,\{adK_1K_2\varepsilon_2^4 + 2adM\varepsilon_2^2 + N_3[(b-d)d + c^2]\} \end{aligned} \tag{2.3}$$

for values of ε_2 in the range $(\varepsilon_2^*, \infty)$, correspond to the sections (η_2^*, η_0) of the axis η_k belonging to the upper half of the wave fields. These lines bound a domain D_2 in the upper θ_2 half-plane, which is symmetric relative to the imaginary axis; the rest of this half-plane is denoted by C_2. Because of symmetry, the correspondence between the lower halves of the wave fields and the lower half-planes of the Riemann surface is the same as for the upper.

The plane θ_1, set in conformity to the relationship (1.2) for $k = 1$, corresponds to the domain A_1 of the quasi-longitudinal wave field. The displacements in this domain are expressed by members of the solution (1.1) defined on the plane θ_1.

The domains D_2 on the plane θ_2, set in correspondence to the relation (1.2) for $k = 2$ correspond to the domain B_1 of the quasi-longitudinal wave field; displacements in these domains are expressed by terms of the solution (1.1) for $k = 2$, defined in the domain D_2.

The domain C_2 in the plane θ_2, set in conformity to the relation (1.2) for $k = 2$ corresponds to the quasi-transverse wave field; displacements are expressed by terms of the solution (1.1) for $k = 2$, defined in the domain C_2.

The features mentioned of determining the solution (1.1) on the Riemann surface are not investigated in /1/; it was assumed that the quasi-longitudinal displacements are expressed by terms of the solution (1.1) defined on the plane θ_1, and the quasi-transverse by terms in the solution (1.1) defined on the θ_2 plane.

For images of the wave pattern on the planes $\xi_k\eta_k$, the wave fields from an instantaneous pulse applied at the point $\xi_k = 0, \eta_k = \eta_0$ are represented by domain bounded by the closed curves (2.1), which are moved to the boundary of the half-space $\eta_k = 0$ with the course of time, without undergoing geometric changes up to reaching the boundary. Central points of the wave fields move along the axes η_k asymptotically approaching the half-space boundary. The line $\eta_k = \eta_0$ divide the wave field into two parts; the points of one of them, for which $\eta_k = \eta_0$, do not meet the half-space boundary, while the points of the other for which $\eta_k \geqslant \eta_0$, reach it in the course of time.

3. At the time $t = y_0 / \sqrt{b}$ the quasi-longitudinal perturbation field reaches the half-space boundary and excites two kinds of reflected waves.

For $\lambda_1 (\theta_2^\circ) y_0 > t > y_0 / \sqrt{b}$, when $\sqrt{b} > \eta_0 > 1 / \lambda_1 (\theta_2^\circ)$, quasi-longitudinal perturbations A_1 characterized by the first terms of the solution (1.1) reach the boundary:

$$u_1 = \mathrm{Re}\, U (\theta_1, \lambda_1, w_1 (\zeta)), \quad v_1 = \mathrm{Re}\, V (\theta_1, \lambda_1, w_1 (\zeta)) \tag{3.1}$$

The variable θ_1 is defined by the relationship

$$1 - \theta_1 \xi_1 + \lambda_1 (\eta_1 - \eta_0) = 0 \tag{3.2}$$

The reflected quasi-longitudinal and quasi-transverse perturbations are expressed by the functions

$$\begin{aligned} u_{11} &= \mathrm{Re}\, U \left(\theta_{11}, \lambda_1, \frac{R_1 (\zeta)\, w_1 (\zeta)}{R (\zeta)(\lambda_1 - \lambda_2)}\right) \\ v_{11} &= \mathrm{Re}\, V \left(\theta_{11}, \lambda_1, \frac{R_1 (\zeta)\, w_1 (\zeta)}{R_1 (\zeta)(\lambda_1 - \lambda_2)}\right) \\ u_{12} &= \mathrm{Re}\, U \left(\theta_{12}, \lambda_2, \frac{A (\zeta; \lambda_1)\, w_1 (\zeta)}{R (\zeta)(\lambda_1 - \lambda_2)}\right) \\ v_{12} &= \mathrm{Re}\, V \left(\theta_{12}, \lambda_2, \frac{A (\zeta; \lambda_1)\, w_1 (\zeta)}{R (\zeta)(\lambda_1 - \lambda_2)}\right) \end{aligned} \tag{3.3}$$

The variables θ_{11} and θ_{12} are defined by the relations

$$1 - \theta_{11} \xi_{11} - \lambda_1 (\eta_{11} + \eta_0) = 0, \quad 1 - \theta_{12} \xi_{12} - \lambda_2 \eta_{12} - \lambda_1 \eta_0 = 0 \tag{3.4}$$

Let us note that errors were made in the expressions (19) in /1/ that correspond to (3.4); the factor $\lambda_1 W_1 (\theta) / (\lambda_1 - \lambda_2)$ is omitted in the right side of (2.3).

The relationship (3.2) sets the segment γ_1, which is symmetric with respect to the imaginary axis, intersects this latter at the point $\theta_1 < \theta_2^\circ$, and approaches points of the upper edge of the slit $(-1 / \sqrt{a}, +1 / \sqrt{a})$ by the ends, into correspondence with the section of the half-space boundary on the upper half-plane θ_1, which the quasi-longitudinal perturbations (3.1) reached. The domain bounded by the segment γ_1 and the section of the upper edge of the slit is denoted by D_1. Points of the incident quasi-longitudinal wave field that passed the half-space boundary correspond to points of the domain D_1, while the section of the incident wave front that has passed the boundary corresponds to the section of the upper edge of the slit between the tip points of the segment γ_1. The solution (3.1) loses physical meaning at points of the domain D_1. The incident waves are expressed at the time under consideration by functions (1.1) defined on the part of the Riemann surface that does not contain the domain D_1.

A point on the quasi-longitudinal and quasi-transverse reflected wave fields, to which the relations (3.4) set the complex points θ_{11} and θ_{12} in correspondence, will correspond to each point of the incident wave field that has passed the half-space boundary. The points θ_{11} and θ_{12} occupy the same domain in the complex plane as does the point θ_1 corresponding to points of the incident wave field which has passed the half-space boundary. On the half-space boundary $\theta_1 = \theta_{11} = \theta_{12}$. It can be considered that as the incident wave field points cross the half-space boundary, their corresponding points θ_1 split into θ_{11} and θ_{12}, to form a double-layer domain D_1 on the upper θ_1 half-plane. The relationship (3.4) sets points of the quasi-longitudinal reflected wave field in correspondence to the points θ_{11} of one layer, and points of the quasi-transverse reflected wave field in correspondence with the points θ_{12} of the other layer. At the times under consideration, the quasi-longitudinal and quasi-transverse reflected waves are expressed by the functions (3.3) determined in the two-layered domain D_1 which broaden in the course of time.

The quasi-longitudinal and quasi-transverse reflected wave fronts are expressed as envelopes of the lines (3.4) for real values of θ_{11}, θ_{12} and λ_k in the form

$$\begin{gathered} \xi_{11} = -\lambda_1' / (\lambda_1 - \theta_{11} \lambda_1'), \quad \eta_{11} = 1 / (\lambda_1 - \theta_{11} \lambda_1') - \eta_0 \\ \xi_{12} = -\lambda_2' \eta_{12} - \lambda_1' \eta_0, \quad \eta_{12} = [1 - (\lambda_1 - \theta_{12} \lambda_1') \eta_0] / (\lambda_2 - \theta_{12} \lambda_2') \end{gathered} \tag{3.5}$$

and are determined by a section of the upper edge of the slit $(-1 / \sqrt{a}, +1 / \sqrt{a})$ enclosed between the ends of the segment γ_1.

At the time $t = \lambda_2 (\theta_2^\circ) y_0$ the quasi-longitudinal perturbations of the domain B_1, expressed by the second terms in the solution (1.1) defined in the domain D_2 of the plane θ_2, reach the half-space boundary

$$u_1^* = \operatorname{Re} U(\theta_2, \lambda_2, w_2(\zeta)), \quad v_1^* = \operatorname{Re} V(\theta_2, \lambda_2, w_2(\zeta)) \tag{3.6}$$

The variable θ_2 is defined by the relationship

$$1 - \theta_2\xi_1 + \lambda_2(\eta_1 - \eta_0) = 0 \tag{3.7}$$

The quasi-longitudinal and quasi-transverse reflected perturbations caused by the perturbations (3.6) are expressed by the functions

$$\begin{aligned} u_{11}^* &= \operatorname{Re} U\left(\theta_{21}, \lambda_2 \frac{R_1(\zeta)\, w_2(\zeta)}{R(\zeta)(\lambda_2 - \lambda_1)}\right) \\ v_{11}^* &= \operatorname{Re} V\left(\theta_{21}, \lambda_2, \frac{R_1(\zeta)\, w_2(\zeta)}{R(\zeta)(\lambda_2 - \lambda_1)}\right) \\ u_{12}^* &= \operatorname{Re} U\left(\theta_{22}, \lambda_1, \frac{A(\zeta; \lambda_2)\, w_2(\zeta)}{R(\zeta)(\lambda_2 - \lambda_1)}\right) \\ v_{12}^* &= \operatorname{Re} V\left(\theta_{22}, \lambda_1, \frac{A(\zeta; \lambda_2)\, w_2(\zeta)}{R(\zeta)(\lambda_2 - \lambda_1)}\right) \end{aligned} \tag{3.8}$$

The variables θ_{21} and θ_{22} are defined by the relationships

$$\begin{aligned} 1 - \theta_{21}\xi_{11}^* - \lambda_2(\eta_{11}^* + \eta_0) &= 0 \\ 1 - \theta_{22}\xi_{12}^* - \lambda_1\eta_{12}^* - \lambda_2\eta_0 &= 0 \end{aligned} \tag{3.9}$$

The functions (3.8) have the same form as the solution (32) in /1/, however, the latter expresses the quasi-transverse and quasi-longitudinal reflected perturbations excited by a quasi-transverse incident wave. Let us note that there is an error in the expression for V_4 in /1/: the factor $R_1(\zeta) / R(\zeta)$ has been omitted in the integrand.

For $\lambda_2(\theta_2^*)y_0 > t > \lambda_2(\theta_2^\circ)y_0$, when $1 / \lambda_2(\theta_2^\circ) > \eta_0 > 1 / \lambda_2(\theta_2^*)$, part of the domain B_1 of the incident quasi-longitudinal wave field crosses the half-space boundary, but the point η_2^* of this domain still does not reach the boundary.

The relationship (3.2) sets segments $\gamma_1^{(1)}$ on the upper θ_1 half-plane from the edge of the slit $(+\theta_2^\circ, \infty)$ to the upper edge of the slit $(-1 / \sqrt{a}, +1 / \sqrt{a})$, and located symmetrically relative to the imaginary axis, in correspondence with sections of the half-space boundary with which quasi-longitudinal perturbations of the domain A_1 make contact. The domains $D_1^{(1)}$ bounded by the segments $\gamma_1^{(1)}$ and the sections of the upper edge of the slit, correspond to points of the domain A_1 which have crossed the half-space boundary.

The relationships (3.4) set points of the domains A_{11} and A_{12} of the quasi-longitudinal and quasi-transverse reflected perturbations expressed by the functions (3.3), into correspondence with the points θ_{11} and θ_{12} of the double-layer domain $D_1^{(1)}$.

The relationship (3.7) sets the line $\gamma_2^{(1)}$ in the domain D_2 on the upper half-plane θ_2, which is a continuation of the segment $\gamma_1^{(1)}$ in the plane θ_1, into correspondence with the section of the half-space boundary which the quasi-longitudinal perturbations of the domain B_1, expressed by the functions (3.6), have reached. The domains $D_2^{(1)}$, bounded by the lines $\gamma_2^{(1)}$, correspond to points of the domain B_1 that have crossed the half-space boundary.

The relationships (3.9) set points of the domains A_{11}^* and A_{12}^* of the quasi-longitudinal and quasi-transverse reflected perturbations expressed by the functions (3.8) into correspondence with the points θ_{21} and θ_{22} of the double-layer domain $D_2^{(1)}$.

The functions λ_1 and λ_2 take on identical values on opposite edges of the slits $(+\theta_2^\circ, \infty)$ of the θ_1 and θ_2 planes. The first relations in (3.4) and (3.9) set the boundary between the domains A_{11}^* and A_{11}^* of the quasi-longitudinal reflected wave field in correspondence with sections of these slit edges included between the points θ_2° and the ends of the lines $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$, and the second relations of (3.4) and (3.9) set the boundary between the domains A_{12} and A_{12}^* of the quasi-transverse reflected wave field in correspondence.

If $t > \lambda_2(\theta_2^*)y_0$, then $1 / \lambda_2(\theta_2^*) > \eta_0$ and the point η_2^* of the domain B_1 of the quasi-longitudinal incident wave field will cross the half-space boundary. As in the preceding case, the reflected waves are described by the functions (3.3) and (3.8), however, the domain $D_1^{(1)}$ on the plane θ_1 will be bounded by the lines $\gamma_1^{(2)}$, and the domain $D_2^{(1)}$ on the plane θ_2 by the lines $\gamma_2^{(2)}$ and the section of the line L_2 included between the lines $\gamma_2^{(2)}$.

4. At the time $t = y_0 / \sqrt{d}$ quasi-transverse perturbations expressed by the second terms of the solution (1.1) defined in the domain C_2 of the plane θ_2 reach the half-space boundary:

$$\begin{aligned} u_2 &= \operatorname{Re} U(\theta_2, \lambda_2, w_2(\zeta)) \\ v_2 &= \operatorname{Re} V(\theta_2, \lambda_2, w_2(\zeta)) \end{aligned} \tag{4.1}$$

The variable θ_2 is defined by the relation

$$1-\theta_2\xi_2+\lambda_2(\eta_2-\eta_0)=0 \tag{4.2}$$

The reflected quasi-longitudinal and quasi-transverse perturbations caused by the perturbations (4.1) are expressed by the functions

$$\begin{aligned} u_{21}&=\operatorname{Re}U\left(\theta_{21},\lambda_1,\frac{A(\zeta;\lambda_2)\,w_2(\zeta)}{R(\zeta)(\lambda_2-\lambda_1)}\right)\\ v_{21}&=\operatorname{Re}V\left(\theta_{21},\lambda_1,\frac{A(\zeta;\lambda_2)\,w_2(\zeta)}{R(\zeta)(\lambda_2-\lambda_1)}\right)\\ u_{22}&=\operatorname{Re}U\left(\theta_{22},\lambda_2,\frac{R_1(\zeta)\,w_2(\zeta)}{R(\zeta)(\lambda_2-\lambda_1)}\right)\\ v_{22}&=\operatorname{Re}V\left(\theta_{22},\lambda_2,\frac{R_1(\zeta)\,w_2(\zeta)}{R(\zeta)(\lambda_2-\lambda_1)}\right)\end{aligned} \tag{4.3}$$

The variables θ_{21} and θ_{22} are defined by the relations

$$\begin{aligned}1-\theta_{21}\xi_{21}-\lambda_1\eta_{21}-\lambda_2\eta_0&=0\\ 1-\theta_{22}\xi_{22}-\lambda_2(\eta_{22}+\eta_0)&=0\end{aligned} \tag{4.4}$$

For $\lambda_2(\theta_2^*)y_0>t>y_0/\sqrt{d}$ when $\sqrt{d}>\eta_0>1/\lambda_2(\theta_2^*)$, part of the incident quasi-transverse wave field will cross the half-space boundary, and the point η_2^* of this field will not reach the boundary.

The relationship (4.2) sets a line β_2 in the domain C_2 of the upper θ_2 half-plane intersecting,the imaginary axis within the section $(0,\theta_2^*)$, into correspondence with a section of the half-space boundary which the perturbations (4.1) reached. For $t=\lambda_2(\theta_2^*)y_0$ the point η_2^* of the quasi-transverse perturbations reaches the half-space boundary, and the line β_2 intersects the imaginary axis at the point θ_2^*. Part of the incident quasi-transverse wave field that has crossed the half-space boundary corresponds to the domain $C_2^{(1)}$ bounded by the line β_2 and a section of the upper edge of the slit $(-1/\sqrt{d},+1/\sqrt{d})$.

For $t>\lambda_2(\theta_2^*)y_0$, when $1/\lambda_2(\theta_2^*)>\eta_0$, the point η_2^* of the quasi-transverse incident wave field will cross the half-space boundary. The relationship (4.2) sets the line $\beta_2^{(1)}$ in the domain C_2 of the upper θ_2 half-plane, that goes from the upper edge of the slit $(-1/\sqrt{d},+1/\sqrt{d})$ to the line L_2 and is symmetric relative to the imaginary axis, in correspondence with the section of the half-space boundary that the quasi-transverse perturbations have reached. The points $\eta_2\geqslant\eta_2^*$ of the quasi-transverse wave field on the axis η_2 that have crossed the half-space boundary correspond to the section of the line L_2 included between the ends of the line $\beta_2^{(1)}$. In contrast to the preceding case, the domain $C_2^{(1)}$ is bounded by the lines $\beta_2^{(1)}$ and sections of the line L_2 and the upper edge of the slit $(-1/\sqrt{d},+1/\sqrt{d})$.

The solution (4.1) becomes a meaningless physically at points of the domain $C_2^{(1)}$. The relations (4.4) set points of the quasi-longitudinal and quasi-transverse reflected wave fields expressed by the functions (4.3) into correspondence with the points θ_{21} and θ_{22} of the two-layered domain $C_2^{(1)}$.

The coefficients of $w_2(\zeta)$ in (4.3) contain the function λ_1 which has real values in the interval $(-1/\sqrt{a},+1/\sqrt{a})$ and imaginary in the range $(\pm1/\sqrt{a},\pm1/\sqrt{d})$.

If points of the section of the boundary of the two-layered domain $C_2^{(1)}$ belonging to the upper edge of the slit $(-1/\sqrt{d},+1/\sqrt{d})$ satisfy the condition $-1/\sqrt{a}\leqslant\theta_{2i}\leqslant+1/\sqrt{a}$, the coefficients of $w_2(\zeta)$ take on real values, and the solutions (4.3) vanish.

The quasi-longitudinal and quasi-transverse reflected wave fronts agree with the envelopes of the lines (4.4), and are, on the section of the boundary of the two-layered domain $C_2^{(1)}$ belonging to the upper edge of the slit, expressed by the functions

$$\begin{aligned}\xi_{21}&=-\lambda_1'\eta_{21}-\lambda_2'\eta_0,\quad \eta_{21}=[1-(\lambda_2-\theta_{21}\lambda_2')\eta_0]/(\lambda_1-\theta_{21}\lambda_1')\\ \xi_{22}&=-\lambda_2'/(\lambda_2-\theta_{22}\lambda_2'),\quad \eta_{22}=1/(\lambda_2-\theta_{22}\lambda_2')-\eta_0\end{aligned} \tag{4.5}$$

When the boundary section of the domain $C_2^{(1)}$ belonging to the upper edge of the slit overlaps the interval $(-1/\sqrt{a},+1/\sqrt{a})$ to emerge beyond its boundary, the coefficients for $w_2(\zeta)$ take on complex values at the points θ_{21} and θ_{22} of the intervals $(\pm1/\sqrt{a},\pm1/\sqrt{d})$, and the solutions (4.3) do not vanish.

At the points θ_{21} the first relationship of (4.4) decomposes into two equations

$$1-\theta_{21}\xi_{21}-\lambda_2\eta_0=0,\ \eta_{21}=0 \tag{4.6}$$

i.e., the reflected quasi-longitudinal perturbations are propagated along the half-space boundary without penetrating into the bulk.

Sections of the envelope of the line expressed by the second relation in (4.4) do not agree with the front of the reflected quasi-transverse perturbations for values of θ_{22} from the interval $(\pm 1/\sqrt{a}, \pm 1/\sqrt{d})$ since the values of the third and fourth expressions in (4.3) are not zero there. In order to satisfy the boundary conditions, we continue the solution expressed by the third and fourth expressions of (4.3) along the semi-tangents to sections of the envelope of the line (4.4) for values of θ_{22} from the intervals $(\pm 1/\sqrt{a}, \pm 1/\sqrt{d})$.

These sections of the envelope with the plane $\eta_2 = 0$ and the semi-tangents corresponding to the values $\theta_{22} = \pm 1/\sqrt{a}$, limit the domain of indirect perturbations. Points of the quasi-transverse wave front (4.1) corresponding to the values θ_2 in the interval $(\pm 1/\sqrt{a}, \pm 1/\sqrt{d})$ cause reflected quasi-longitudinal perturbations by reaching the half-space boundary which in turn excite a quasi-transverse wave by being propagated along the boundary more rapidly than the quasi-transverse wave that leads the quasi-transverse wave being reflected by the usual law.

In this case, the front of the reflected quasi-transverse wave is determined by the second expressions of (4.5) in the range $-1/\sqrt{a} \leqslant \theta_{22} \leqslant +1/\sqrt{a}$ and by the two semi-tangents corresponding to the values $\theta_{22} = \pm 1/\sqrt{a}$, the front of the reflected quasi-longitudinal wave is determined by the first expressions in (4.5) in the interval $-1/\sqrt{a} \leqslant \theta_{21} \leqslant +1/\sqrt{a}$.

In conclusion, let us note that in contrast to wave propagation in an isotropic medium, the wave process in an anisotropic half-space must be considered as a single complex process consisting of primary quasi-longitudinal and quasi-transverse perturbations being propagated from a source, and of secondary quasi-longitudinal and quasi-transverse perturbations reflected from the half-space boundary.

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